Linear codes over $Z_4+uZ_4+u^2Z_4$ and MacWilliams Identities

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Abstract

Linear codes are considered over the ring $Z_4+uZ_4+u^2Z_4$ Lee weights, Gray maps for these codes are defined and MacWilliams identities for the complete, symmetrized and Lee weight enumerators are proved.

Key words: MacWilliams identities, Weight enumerators ,Gray map

1. Introduction

Linear codes are important part of coding theory and play a key role in error correction. They are more effective than other codes because they encode and decode rapidly. Thus they are most preferred codes because of their algebraic properties. Another important part of coding theory is the MacWilliams identity which relates the weight enumerator of linear code to the weight enumerator of its dual code. In [2], MacWilliams identity for ρ weight enumerator over linear codes in matrices is proved.In [1], MacWilliams identity for ρ complete weight enumerator of matrices with elements from ring $F_q[u]/(u^r-a)$ where F_q denotes a finite field with q elements and $a\in F_q$ is proved. Then, it is endowed to [3]. In [4] linear code are investigated over Z_4+uZ_4 and MacWilliams identities for a variety of weight enumerators are proved. In [5] Linear codes are studied over the ring $F_p+uF_p+u^2F_p$, where p is an odd prime. Also,Gray map and MacWilliams identity of linear codes are given.

This paper is organized as follows: In Section II we are considered linear codes over the ring $Z_4+uZ_4+u^2Z_4=\{a+ub+u^2c\,|\,a,b,c\in Z_4\}$ with $u^3=u$ and we give definitions of Gray map, Lee distance, and dual code of linear code. In Section III we have studied complete, symmetrized and Lee weight enumerators and we proved MacWilliams identities for these weight enumerators.

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2. Lee Weight and Gray Image of Linear Codes over $Z_4 + uZ_4 + u^2 Z_4$

Definition 1: A linear code C of length n over the ring $Z_4 + uZ_4 + u^2 Z_4$ is defined by on $Z_4 + uZ_4 + u^2 Z_4$-submodule of $(Z_4 + uZ_4 + u^2 Z_4)^n$.

Definition 2: Let $x = a + ub + u^2 c$ be an element of $Z_4 + uZ_4 + u^2 Z_4$. Then the Lee weight of x is defined as follows;

$$w_L(x) = w_L((a, a+b+c, b))$$

Where $w_L((a, a+b+c, b)$ denotes the Lee weight on Z_4^3. Lee Weight of a codeword $x = (x_0, x_1, ..., x_{n-1}) \in R^n$ is defined by rational sum of the Lee weight of its components; i.e. $w_L(x) = \sum_{i=0}^{n-1} w_L(x_i)$. For any $x, y \in R^n$, the Lee distance is given by $d_L(x, y) = w_L(x - y)$. The minimum Lee distance of C is the smallest nonzero Lee distance between all pairs of distinct codewords. The minimum Lee weight of C is the smallest nonzero Lee weight among all codewords. If C is linear, then minimum Lee weight is equal to minimum Lee distance.

Definition 3: The Gray map φ on $Z_4 + uZ_4 + u^2 Z_4$ is given by

$$\varphi : Z_4 + uZ_4 + u^2 Z_4 \to Z_4^3$$
$$a + ub + u^2 c \to (a, a+b+c, b)$$

where $a, b, c \in Z_4$. It is easy to verify that the Lee weight of C in $Z_4 + uZ_4 + u^2 Z_4$ is the Lee weight of $\varphi(C)$ in Z_4.

Theorem 4: The Gray map φ is a distance preserving linear isometry from ($Z_4 + uZ_4 + u^2 Z_4$, Lee distance) to (Z_4, Lee distance).

Corollary 5: If C is linear code over $Z_4 + uZ_4 + u^2 Z_4$ of length n , then $\varphi(C)$ linear code over Z_4 of length 3*n.*

2.3. The Dual of Linear Code

The inner product of $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0 y_1, ..., y_{n-1})$ in R^n defined by;

$$< x.y >= \sum_{i=0}^{n-1} x_i y_i$$

where the operations are performed in $Z_4+uZ_4+u^2Z_4$.

Definition 7: If C is a linear code over $Z_4+uZ_4+u^2Z_4$ of length n , then the dual code of C is defined as;

$$C^{\perp}=\{x\in(Z_4+uZ_4+u^2Z_4)^n :< x.y >= 0;\forall y\in C\}$$

3. Weight Enumerators and MacWilliams Identities

We label all the elements of $Z_4+uZ_4+u^2Z_4$ with n_i symbol for $1\le i\le 64$.

Let $Z_4+uZ_4+u^2Z_4=\{n_1,n_2,...,n_{64}\}$ be given as

$$\begin{aligned}Z_4+uZ_4+u^2Z_4=\{&0,1,2,3,u,2u,3u,u^2,2u^2,3u^2,1+u,1+2u,1+3u,1+u^2,1+2u^2,1+3u^2,2+u,\\&2+2u,2+3u,2+u^2,2+2u^2,2+3u^2,3+u,3+2u,3+3u,3+u^2,3+2u^2,3+3u^2,u+u^2,u+2u^2,\\&u+3u^2,2u+u^2,2u+2u^2,2u+3u^2,3u+u^2,3u+2u^2,3u+3u^2,1+u+u^2,1+u+2u^2,1+u+3u^2,\\&1+2u+u^2,1+2u+2u^2,1+2u+3u^2,1+3u+u^2,1+3u+2u^2,1+3u+3u^2,2+u+u^2,2+u+2u^2,\\&2+u+3u^2,2+2u+u^2,2+2u+2u^2,2+2u+3u^2,2+3u+u^2,2+3u+2u^2,2+3u+3u^2,\\&3+u+u^2,3+u+2u^2,3+u+3u^2,3+2u+u^2,3+2u+2u^2,3+2u+3u^2,3+3u+u^2,3+3u+2u^2,\\&3+3u+3u^2\}\end{aligned}$$

Definition 8: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$ and n_i is any element of R. For all $c=(c_0,c_1,...,c_{n-1})\in R^n$ defined weight of c at a to be

$$s_{n_i}(c)=\left|\{j:c_j=n_i\}\right|$$

The complete weight enumerator of a linear code over $Z_4+uZ_4+u^2Z_4$ is defined as;

$$cwe_C(X_1,X_2,...,X_{64})=\sum_{c\in C}X_1^{s_{n_1}(c)}X_2^{s_{n_2}(c)}...X_{64}^{s_{n_{64}}(c)}$$

Since some elements have some Lee weights, then we can define symmetrized weight enumerator as follows.

Definition 9: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$. Then define the symmetrized weight enumerator of C as;

$swe_C(X_0,X_1,X_2,X_3,X_4,X_5,X_6)=cwe_C(X_0,X_2,X_4,X_2,X_2,X_4,X_2,X_1,X_2,X_1,X_4,X_4,X_2,$
$X_3,X_2,X_1,X_4,X_4,X_4,X_3,X_2,X_3,X_2,X_4,X_4,X_1,X_2,X_3,X_3,X_2,X_1,X_3,X_2,X_3,X_1,X_2,$
$X_3,X_3,X_2,X_3,X_3,X_4,X_5,X_3,X_4,X_3,X_3,X_4,X_5,X_5,X_6,X_5,X_5,X_4,X_3,X_3,X_4,X_3,X_5,$
$X_4,X_3,X_3,X_2,X_3)$

Here X_i denotes the elements that have weight i where $0 \le i \le 1$.

Definition 10: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$. Then Lee weight enumerator of C is given by

$$Lee_C(X_0,X_1)=\sum_{c\in C} X_1^{6n-w_L(c)} X_0^{w_L(c)}$$

Let $c\in C$ be a codeword. If $\alpha_i(c)$ denotes the number of elements of c with Lee weight i, then we obtain some results as follows;

i) $swe_C(X_0,X_1,X_2,X_3,X_4,X_5,X_6)=\sum_{c\in C} X_0^{\alpha_0(c)}X_1^{\alpha_1(c)}X_2^{\alpha_2(c)}X_3^{\alpha_3(c)}X_4^{\alpha_4(c)}X_5^{\alpha_5(c)}X_6^{\alpha_6(c)}$

ii) $w_L(c)=\alpha_1(c)+2\alpha_2(c)+3\alpha_3(c)+4\alpha_4(c)+5\alpha_5(c)+6\alpha_6(c)$

Theorem 11: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$. Then

$$Lee_C(X_0,X_1)=swe_C(X_1^6,X_1^5X_0,X_1^4X_0^2,X_1^3X_0^3,X_1^2X_0^4,X_1X_0^5,X_0^6)$$
$$Lee_C(X_0,X_1)=Lee_{\varphi(C)}(X_0,X_1)$$

Now we define MacWilliams identities for complete, Lee and symmetrized weight enumerator. To find the identities we define the following character on $Z_4+uZ_4+u^2Z_4$

Definition 12: Let I be non-zero ideal of $Z_4+uZ_4+u^2Z_4$. Define $\chi: I\to\mathbb{C}^*$ by $\chi(a+ub+u^2c)=i^c$ where $\mathbb{C}^*$ is the multiplicative group of unit complex number. χ is a non-trivial character of I and hence we have $\sum_{a\in I}\chi(a)=0$.

Theorem 13: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$ and suppose $C^\perp$ is its dual. Then

$$cwe_{C^\perp}(X_1,X_2,...,X_{64})=\frac{1}{|C|}cwe_C(G.(X_1,X_2,...,X_{64})^T)$$

where G is an 64 x 64 matrices defined by $G(i,j)=\chi(n_in_j)$.

Theorem 14: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$ and suppose $C^\perp$ is its dual. Then

$$swe_{C^{\perp}}(X_0,X_1,X_2,X_3,X_4,X_5,X_6)=\frac{1}{|C|}swe_C(X_0+6X_1+15X_2+20X_3+15X_4+6X_5+X_6;$$
$$X_0+4X_1+5X_2-5X_4-4X_5-X_6;X_0+2X_1-X_2-4X_3-X_4+2X_5+X_6;$$
$$X_0-3X_2+3X_4-X_6;X_0-2X_1-X_2+4X_3-X_4-2X_5+X_6;$$
$$X_0-4X_1+5X_2-5X_4+4X_5-X_6;X_0-6X_1+15X_2-20X_3+15X_4-6X_5+X_6)$$

Proof: The proof follows from Theorem 13 and the definition of symmetrized weight enumerator

Theorem 15: Let C be a linear code of length n over $Z_4+uZ_4+u^2Z_4$ and suppose $C^{\perp}$ is its dual. Then

$$Lee_C(X_0,X_1)=\frac{1}{|C|}Lee_{C^{\perp}}(X_0+X_1,X_0-X_1)$$

Proof : The proof follows from Theorem 11 and Theorem 14.

Conclusions

In this paper we investigate linear codes over the ring $Z_4+uZ_4+u^2Z_4$. Lee weights, Gray maps for these codes are defined and MacWilliams identities for the complete, symmetrized and Lee weight enumerators are proved.

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