Finite Time Blow Up Of Solutions To Fourth-Order Equation With Power Nonlinearity

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Abstract

In this paper we consider a fourth order nonlinear parabolic equation with power type nonlinearity which such types of problems occur in many mathematical models of applied science, such as chemical reactions, heat transfer, population dynamics, electro-rheological fluids etc. We show that there are solutions under some conditions on initial data which blow up in finite time with positive initial energy. To show the nonexistence of solutions under initial functions we used to generalized concavity lemma.

Keywords: Finite time blow up, fourth order equation, generalized concavity.

1 Introduction

Many physical and engineering problems can be modelled mathematically in the form of evolution equations (partial differential equations depending on time). We can not obtain a well-defined solution for these equations without adding suitable additional conditions (initial and boundary conditions). Since the last century, many authors have studied the existence and uniqueness for the linear types of these problems.

Nonlinear partial differential equations are more complicated and have more properties than linear equations, these properties are related to important features of the real world phenomena, on the other hand, these properties are connected with the difficulties of the mathematical treatment.

In the last decades, partial differential equations became one of the most active areas of mathematics research because it helped mathematicians to find answers and explanations to many phenomena of the nonlinear world.

It is known that singularities occur in the solution of linear problems when the problem has singular coefficients or singular data, the so called fixed singularities. One of the most important properties of nonlinear partial differential equations is the possibility of eventual occurrence of singularities starting from smooth data (coefficient and initial or boundary conditions), the so called well posedness in the small, meaning the existence and uniqueness and continuity of the classical solutions can be established for small time.
Singularities of nonlinear problems may come from the effects of nonlinear terms, which occur in the partial differential equations or in the boundary conditions, usually they depend on the time and the location, the so called moving singularities. One of the most remarkable type of these

singularities is what we call the Blow-up phenomena. Basically, in a nonlinear problem, blow-up is a form of the spontaneous singularities appear when one or more of the depending variables go to infinity as time goes to a certain finite time.

We consider the following fourth order quasilinear parabolic equation :

$$u_t - \Delta[(k_0 + k_1|\Delta u|^{m-2})\Delta u)] - g(x,t,u,\Delta u) = |u|^{p-2}u \quad (1)$$
$$u(x,t) = \Delta u = 0,\ x \in \partial\Omega, t > 0 \quad (2)$$
$$u(x,0) = u_0,\ x \in \Omega \quad (3)$$

where $\Omega \subseteq \mathbb{R}^n, n \geq 1$ is bounded domain with a sufficiently smooth boundary $\partial\Omega$. Also the constants $k_0\ and\ k_1$ are positive numbers and $p > m+1 > 3$. Also assume that $u_0(x)$ is given function satisfying

$$u_0 \in H_0^2(\Omega) \cap L^p(\Omega) \quad (4)$$

and $g(x,t,u,\Delta u)$ is continuous function which have the relation

$$|g(x,t,u,\Delta u)| \leq M\left(|u|^{\frac{p}{2}} + |\Delta u|^{\frac{m}{2}}\right) \quad (5)$$

with some positive $M > 0$.

Existence of solutions to these type of equations are studied in [1]. Erdem, in [2], studied blow-up solutions to quasilinear parabolic equations

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left((d + |\nabla u|^{m-2}), \frac{\partial u}{\partial x_i}\right) + g(u,\nabla u) = f(u)$$

where d is positive constant and f and g are continuous functions which satisfy the following conditions;

$$(u,(f(u))) \geq 2(1+\alpha)G(u), \alpha > 0,\ G(u) = \int_0^u f(s)ds\ ,\ |g(u,v)| \leq c_1(|u| + |v|)\ , c_1 > 0$$

Zhou, in [4], considered the following quasilinear parabolic equation

$$a(x,t)u_t - \mathrm{div}(|\nabla u|^{m-2}\nabla u) = f(u)$$

where $a(x,t)u \geq 0$ is a generalized Lewis function. He obtain blow-up result in finite time if the initial data possesses suitable positive energy.

Shahrouzi, [5], investigated a fourth order nonlinear wave equation with dissipative boundary condition. He showed that there was solutions under some conditions on initial data which blowed-up in finite time with positive initial energy.

In this work, we consider blow up results in finite time for solutions to quasilinear parabolic equation (1)-(3) with positive initial energy.

Thoroughout this paper, we use the following notations;

$\|u\| = \|u\|_{L_2(\Omega)}$, $\|u\|_{\mathrm{p}} = \|u\|_{L_p(\Omega)}$ are usual the lebesque spaces, $(u, v) = \int_{\Omega} uvdx$ is the inner product,

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \tag{6}$$

is the weighted arithmetic-geometric inequality for $a, b > 0$ and

$$ab \leq \beta a^q + C(p,\beta) b^{q'} \tag{7}$$

is the Young`s inequality with $\frac{1}{q} + \frac{1}{q'} = 1, C(q,\beta) = \frac{1}{q'(q\beta)^{q'/q}}$.

Let us note the following lemma known as generalized concavity lemma or "Ladyzhenskaya-Kalantarov lemma". It is good tool to obtain the blow up results for dynamical problems.

Lemma 1 *Suppose that a positive, twice differentiable function* $\psi(t)$ *satisfies for* $t > 0$ *the following inequality*

$$\psi\psi'' - (1+\gamma)(\psi')^2 \geq -2M_1\psi\psi' - M_2\psi^2$$

where $\gamma > 0$, $M_1, M_2 \geq 0$. If $(0) > 0$, $\psi'(0) > -\gamma_2\gamma^{-1}\psi(0)$, and $M_1 + M_2 > 0$, then $\psi(t)$ *tends to infinity* as $t \to t_1 \leq t_2$.

$$t_2 \leq \frac{1}{2\sqrt{M_1^2+\gamma M_2}} ln \frac{\gamma_1\psi(0)+\gamma\psi'(0)}{\gamma_2\psi(0)+\gamma\psi'(0)}$$

where $\gamma_1 = -M_1 + \sqrt{M_1^2 + \gamma M_2}$, $\gamma_2 = -M_1 - \sqrt{M_1^2 + \gamma M_2}$.

Proof (see [3])

2 Main Result

Theorem 1 *Suppose that the condition (3) is satisfied. Let* $u(x,t)$ *be the solution of the problem (1)-(3). Assume the following conditions are valid:*

$$\gamma = \sqrt{1+\beta} - 1,\ \beta\epsilon(0,\alpha),\ \alpha = \frac{p+m-4}{8},$$

$$max\left\{\frac{M^2(m+k_1p)}{k_1(p-m)}, \frac{M^2(m+pk_1)(1+\alpha)}{k_1(p-m)(\alpha-\beta)}\right\} < \lambda < \frac{\gamma(p+m)j(0)}{2(1+\gamma)^2\|u_0\|^2} \tag{8}$$

where

$$j(0) = -\frac{\lambda}{2}\|u_0\|^2 - \frac{k_0}{2}\|\Delta u_0\|^2 - \frac{k_1}{m}\|\Delta u_0\|_m^m + \frac{1}{p}\|u_0\|_p^p > 0 \tag{9}$$

Then there exists a finite time t_1 *such that*

$$\|u\|^2 \to +\infty \ \ as\ \ t \to t_1^-.$$

Proof: For $\lambda > 0$, we make the transformation $u(x,t) = e^{\lambda t}v(x,t)$ in (1) and we obtain the equation

$$v_t + \lambda v + \Delta\left[\left(k_0 + k_1 e^{\lambda(m-2)t}|\Delta v|^{m-2}\right)\Delta v\right] - e^{-\lambda t}g\left(e^{\lambda t}v, e^{\lambda t}\Delta v\right) = e^{\lambda(p-2)t}|v|^{p-2}v \tag{10}$$

with the boundary condition and the initial condition

$$v(x,t) = 0,\ x \in \partial\Omega, t > 0\ ,\ v(x,0) = u_0,\ x \in \Omega\ , \tag{11}$$

respectively.

Let us multiply the equation (10) by v_t in $L^2(\Omega)$, we get the relation

$$\|v_t\|^2 + \frac{d}{dt}\left[\frac{\lambda}{2}\|v\|^2 + \frac{k_0}{2}\|\Delta v\|^2 + \frac{k_1}{m}e^{\lambda(m-2)t}\|\Delta v\|_m^m - \frac{1}{p}e^{\lambda(p-2)t}\|v\|_p^p\right]$$

$$-\frac{\lambda k_1(m-2)}{m}e^{\lambda(m-2)t}\|\Delta v\|_m^m + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p = e^{-\lambda t}\left(g\left(e^{\lambda t}v, e^{\lambda t}\Delta v\right), v_t\right)\ . \tag{12}$$

Using the inequalities (6) and (7) to the term on the right side of (12) with condition (5), we have

$$e^{-\lambda t}\left|\left(g\left(x.t.e^{\lambda t}v, e^{\lambda t}\Delta v\right), v_t\right)\right| \le \epsilon_0 e^{\lambda(m-2)t}\|\Delta v\|_m^m + \epsilon_1 e^{\lambda(p-2)t}\|v\|_p^p + \frac{M^2}{4}\left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_1}\right)\|v_t\|^2 \tag{13}$$

Substituting equation (14) in equation (13) we get the relation

$$\|v_t\|^2 - \frac{d}{dt}j(t) - \frac{\lambda k_1(m-2)}{m} e^{\lambda(m-2)t}\|\Delta v\|_m^m + \frac{\lambda(p-2)}{p} e^{\lambda(p-2)t}\|v\|_p^p$$

$$\leq \epsilon_0 e^{\lambda(m-2)t}\|\Delta v\|_m^m + \epsilon_1 e^{\lambda(p-2)t}\|v\|_p^p + \frac{M^2}{4}\left(\frac{1}{\epsilon_0}+\frac{1}{\epsilon_1}\right)\|v_t\|^2 \quad (14)$$

where $j(t) = \frac{1}{p} e^{\lambda(p-2)t}\|v\|_p^p - \frac{\lambda}{2}\|v\|^2 - \frac{k_0}{2}\|\Delta v\|^2 - \frac{k_1}{m} e^{\lambda(m-2)t}\|\Delta v\|_m^m$.

We rewrite the inequality (14) as

$$\frac{d}{dt}j(t) \geq [\lambda(p-2)-\epsilon_1 p]j(t) + \frac{\lambda}{2}[\lambda(p-2)-\epsilon_1 p]\|v\|^2 + \frac{k_0}{2}[\lambda(p-2)-\epsilon_1 p]\|\Delta v\|^2$$

$$+\left(\frac{k_1}{m}[\lambda(p-m)-\epsilon_1 p] - \frac{\lambda k_1(m-2)}{m} - \epsilon_0\right) e^{\lambda(m-2)t}\|\Delta v\|_m^m + \left\{1 - \frac{M^2}{4}\left(\frac{1}{\epsilon_0}+\frac{1}{\epsilon_1}\right)\right\}\|v_t\|^2 \quad (15)$$

We choose $\epsilon_0 = \frac{a\lambda}{m}$ and $\epsilon_1 = \frac{\lambda(p-m-1)}{p}$ in the inequality (15) to get the estimate

$$\frac{d}{dt}j(t) \geq \lambda(m-1)j(t) + \frac{\lambda^2}{2}(m-1)\|v\|^2 + \frac{\lambda k_0}{2}(m-1)\|\Delta v\|^2$$

$$+\left\{1 - \frac{M^2(m(p-m-1)+pk_1)}{4\lambda k_1(p-m-1)}\right\}\|v_t\|^2 \quad (16)$$

Since $m-1>0$, the second and third terms on the right side of (16) can be omitted to get the inequality

$$\frac{d}{dt}j(t) \geq \lambda(m-1)j(t) + \left\{1 - \frac{M^2(m(p-m-1)+pk_1)}{4\lambda k_1(p-m-1)}\right\}\|v_t\|^2 \ . \quad (17)$$

From the assumption (8) we obtain the relation

$$\frac{d}{dt}j(t) \geq \lambda(m-1)j(t) + \left(\frac{1+\beta}{1+\alpha}\right)\|v_t\|^2 \ . \quad (18)$$

Solving the differential inequality (18) we have

$$j(t) \geq j(0)e^{\lambda(m-1)t} + \left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_s\|^2 ds \ .$$

It is easy to see that $j(t) \geq e^{\frac{\lambda}{2}(m-1)t} j(0) \geq j(0)$ by assumption (9). Thus we obtain a lower bound for $j(t)$

$$j(t) \geq \left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau + j(0) \tag{19}$$

Multiplying the equation (10) by v in $L^2(\Omega)$ we get the relation

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + k_0\|\Delta v\|^2 + k_1 e^{\lambda(m-2)t}\|\Delta v\|_m^m - e^{\lambda(p-2)t}\|v\|_p^p$$
$$= e^{-\lambda t}\left(g\left(x,t,e^{\lambda t}v, e^{\lambda t}\Delta v\right), v\right) \tag{20}$$

Using the inequalities (6) and (7) to the term on the right side of (20) under condition (5), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq -k_0\|\Delta v\|^2 + (1-\epsilon_3)e^{\lambda(p-2)t}\|v\|_p^p - (k_1+\epsilon_2)e^{\lambda(m-2)t}\|\Delta v\|_m^m$$
$$-\left[\lambda + \frac{M^2}{4}\left(\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\right)\right]\|v\|^2 \tag{21}$$

Rewrite the inequality (21) as follows

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq \frac{(p+m)}{2}j(t) + \left[\frac{\lambda(p+m-4)}{4} - \frac{M^2}{4}\left(\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\right)\right]\|v\|^2 + \frac{k_0(p+m-4)}{4}\|\Delta v\|^2$$
$$+\left(\frac{k_2(p-m)}{2m} - \epsilon_2\right)e^{\lambda(m-2)t}\|\Delta v\|_m^m + \left(\frac{p-m}{2p} - \epsilon_3\right)e^{\lambda(p-2)t}\|v\|_p^p \tag{22}$$

Choose $\epsilon_2 = \frac{k_2(p-m)}{2m}$ and $\epsilon_3 = \frac{p-m}{2p}$ and omit third term on the right side of the inequality (22) since $(p+m-4) > 0$, then inequality (22) follows

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq \frac{p+m}{2}j(t) - \frac{M^2(m+k_1p)}{k_1(p-m)}\|v\|^2 \tag{23}$$

Substituting the estimate (19) and $p+m = 4(1+2\alpha)$ in (23) we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq 2(1+2\alpha)\left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau + 2(1+2\alpha)j(0) - \frac{M^2(m+k_1p)}{k_1(p-m)}\|v\|^2 \tag{24}$$

By assumption (8) then it follows from (24)

$$\frac{d}{dt}\|v\|^2 \geq 4(1+2\alpha)\left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau - 2\lambda\|v\|^2 + 4(1+2\alpha)j(0) \tag{25}$$

Now let us introduce the positive function

$$\psi(t) = \int_0^t \|v\|^2 d\tau + C_0 \tag{26}$$

where C_0 is a positive constant will be chosen later. First and second derivatives of (26) as follows

$$\psi'(t) = \|v\|^2 = 2\int_0^t (v, v_\tau) d\tau + \|u_0\|^2 \ , \psi''(t) = \frac{d}{dt}\|v\|^2 \tag{27}$$

Apply the Cauchy-Schwarz inequality and the weighted arithmetic-geometric inequality to get an upper bound for $\psi'(t)$;

$$\begin{aligned} [\psi'(t)]^2 &= 4\left[\int_0^t (v, v_\tau) d\tau + \frac{1}{2}\|u_0\|^2\right]^2 \\ &\leq 4\left[(1+4\epsilon)\left(\int_0^t \|v\|^2 d\tau\right)\left(\int_0^t \|v_\tau\|^2 d\tau\right) + \frac{1}{4}\left(1+\frac{1}{4\epsilon}\right)\|u_0\|^4\right] \end{aligned} \tag{28}$$

Remembering the relations (26)-(28) we can estimate the term $\psi\psi'' - (1+\gamma)(\psi')^2$;

$$\begin{aligned} \psi\psi'' - (1+\gamma)(\psi')^2 \geq\ & 4(1+\beta)\left(\int_0^t \|v\|^2 d\tau\right)\psi + (p+m)j(0)\psi - 2\lambda\|v\|^2\psi \\ & -4(1+\gamma)\left[(1+4\epsilon)\left(\int_0^t \|v\|^2 d\tau\right)\left(\int_0^t \|v_\tau\|^2 d\tau\right) + \frac{1}{4}\left(1+\frac{1}{4\epsilon}\right)\|u_0\|^4\right] \end{aligned} \tag{29}$$

We choose $\epsilon > 0$ such that $max\left\{1+4\epsilon, 1+\frac{1}{4\epsilon}\right\} = \frac{1+\beta}{1+\gamma}$. By assumption (8) and $\psi \geq C_0$ we get the estimation

$$\psi\psi'' - (1+\gamma)(\psi')^2 \geq -2\lambda\psi\psi' + C_0(p+m)j(0) - (1+\gamma)^2\|u_0\|^4 . \tag{30}$$

3 Conclusion

As a result the lemma can be applied if

$$C_0 = \frac{(1+\gamma)^2}{(p+m)j(0)}\|u_0\|^4 . \tag{31}$$

So we have $\psi\psi'' - (1+\gamma)(\psi')^2 \geq -2\lambda\psi\psi'$, with $M_1 = \lambda$, $M_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = -2\lambda$. The conditions of lemma $\psi(0) > 0$ and $\psi'(0) > -\gamma_2\gamma^{-1}\psi(0)$ are satisfied by the positive constant (31) and assumption (8) respectively. Thus solutions to the problem for nonlinear equation (1)-(3) blow up as

$$t \to t_1 \leq \frac{1}{2\lambda} ln \frac{\gamma(p+m)j(0)}{\gamma(p+m)j(0) - 2\lambda(1+\gamma)^2\|u_0\|^2} .$$

For example, take $u_0 = sin\pi x$, $\Omega = (0,1), p = 5, m = 3, k_0 = \frac{1}{1000}, k_1 = \frac{1}{10000}, \lambda = \frac{1}{1000}, \beta = \frac{1}{4}, M = \frac{1}{10000}$, with positive initial energy $j(0) = 0{,}016$ then we calculate the blow up time t=43,5 .

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